

Fans in the Theory of Real Semigroups

II. Combinatorial Theory

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Abstract

In [DP5a] we introduced the notion of fan in the categories of real semigroups and their dual abstract real spectra and developed the algebraic theory of these structures. In this paper we develop the combinatorial theory of ARS-fans, i.e., fans in the dual category of abstract real spectra. Every ARS is a spectral space and hence carries a natural partial order called the *specialization partial order*. Our main result shows that the isomorphism type of a finite fan in the category ARS is entirely determined by its order of specialization. The main tools used to prove this result are: (1) Crucial use of the theory of *ternary semigroups*, a class of semigroups underlying that of RSs; (2) Every ARS-fan is a disjoint union of abstract order spaces (called *levels*); (3) Every level carries a natural involution of abstract order spaces, and (4) The notion of a *standard generating system*, a combinatorial tool replacing, in the context of ARSs, the (absent) tools of combinatorial geometry (matroid theory) employed in the cases of fields and of abstract order spaces.

Introduction

In [DP5a] we introduced a notion of *fan* in each of the dual categories **RS** and **ARS** of real semigroups, and of abstract real spectra, dubbed, respectively, *RS-fans* and *ARS-fans*. The emphasis in [DP5a] was on the algebraic theory of RS-fans. The present paper, a continuation of [DP5a], is devoted to develop the combinatorial theory of ARS-fans, i.e., fans in the category of abstract real spectra.

The Introduction to [DP5a] gives an account of the role of fans in the theories of preordered fields, of quadratic forms, and in real algebraic and analytic geometry.

Our main result in this paper is Theorem 3.11, showing that the isomorphism type of a finite ARS-fan (in the category **ARS**) is entirely determined by its order of specialization as a spectral space¹. The proof of this result relies on a combinatorial machinery that we set up in §§1,2. This machinery also gives detailed information on the structure of ARS-fans under their order of specialization.

In Section 1 we introduce the notion of *level* of an ARS-fan (X, F) . Levels are the pieces $L_I = \{h \in X \mid h^{-1}[0] = I\}$ of the partition of the character space X of the real semigroup F , determined by the ideals I of F . Since F is a RS-fan, its ideals are necessarily prime and saturated ([DP5a], Prop. 1.6 (4), Cor. 3.10 (1)), and the family of them is totally ordered under inclusion ([DP5a], Fact 1.4). By Proposition 5.11 of [DP5a], each level is an abstract space of orders and therefore (by results from [D1], [D2] and [Li]) possesses a structure of combinatorial geometry (matroid). Further, there exist canonical AOS-morphisms linking each level to any level determined by a larger ideal (i.e., a “higher” level); Proposition 1.2 (2).

In the next §2 we exploit the combinatorial geometric structure of the levels to investigate the fine structure of ARS-fans. In Theorem 2.8 we show that multiplication of a character of

¹ For a general reference on spectral spaces, see [DST].

level L_I by any pair of elements $g_1, g_2 \in X$ so that $Z(g_i) := g_i^{-1}[0] \subseteq I$ ($i = 1, 2$) defines an involution of the AOS L_I . These involutions are compatible with the order of specialization between levels induced by inclusion of the determining ideals (2.8 (e)). Further, we prove that these involutions permute certain AOS-subfans of the levels defined by combinatorial conditions (Propositions 2.10 and 2.11). Altogether, the results proved in this section show that the order structure of ARS-fans is subject to strong constraints, illustrated in 2.18.

To prove the isomorphism Theorem 3.11, the combinatorial machinery mentioned above is used together with the notion of a *standard generating system* introduced in 3.4. This notion is a substitute for the combinatorial geometric notions existing in the context of AOSs, but absent in that of ARSs.

Preliminaries. For easy reference we state, without proof, the following simple facts proved in [DP5a] and frequently used below. The axioms defining the notion of a *ternary semigroup* (abbreviated TS) appear in [DP5a], Def. 1.1, and [DP1], § 1, p. 100; X_T denotes the set of TS-homomorphisms of a TS, T , into the TS $\mathbf{3} = \{1, 0, -1\}$ (the TS-*characters* of T).

The first Lemma gives several characterizations of the specialization order of the spectral topology on the character set of a ternary semigroup.

Lemma 0.1 *Let T be a TS, and let $g, h \in X_T$. The following are equivalent:*

- (1) $g \rightsquigarrow h$ (i.e., h is an specialization of g).
- (2) $h^{-1}[1] \subseteq g^{-1}[1]$ (equivalently, $h^{-1}[-1] \subseteq g^{-1}[-1]$).
- (3) $g^{-1}[\{0, 1\}] \subseteq h^{-1}[\{0, 1\}]$.
- (4) $Z(g) \subseteq Z(h)$ and $\forall a \in G (a \notin Z(h) \Rightarrow g(a) = h(a))$.
- (5) $h = h^2g$ (equivalently, $h^2 = hg$). □

We also register the following algebraic characterizations of inclusion and equality of zero-sets of elements of X_T .

Lemma 0.2 *Let T be a TS, and let $u, g, h \in X_T$. Then:*

- (1) $Z(g) \subseteq Z(h) \Leftrightarrow h = hg^2$.
- (2) $Z(g) = Z(h) \Leftrightarrow g^2 = h^2$.
- (3) If $u \rightsquigarrow g, h$, then $Z(g) \subseteq Z(h)$ if and only if $g \rightsquigarrow h$. □

Proposition 0.3 *Let G be a RS-fan. Then:*

- (1) For all elements $g, h \in X_G$ such that $g \rightsquigarrow h$ (hence $Z(g) \subseteq Z(h)$) and every ideal I such that $Z(g) \subseteq I \subseteq Z(h)$ there is $f \in X_G$ such that $g \rightsquigarrow f \rightsquigarrow h$ and $Z(f) = I$.
- (2) For every $g \in X_G$ and every ideal $I \supseteq Z(g)$ there is a (necessarily unique) $f \in X_G$ such that $g \rightsquigarrow f$ and $Z(f) = I$.
- (3) For every ideal I of F there is an $f \in X_G$ such that $Z(f) = I$. □

1 Levels of a ARS-fan

The saturated prime ideals of a real semigroup induce a partition of its character space. The pieces are called *levels*: the level corresponding to a saturated prime ideal I of G is the set of all $g \in X_G$ such that $Z(g) = I$. In the case of RS-fans, (proper) ideals —automatically prime and saturated ([DP5a], Prop. 1.6 (4), Cor. 3.10 (1))— are totally ordered under inclusion ([DP5a], Fact 1.4), a fact of much help in studying the relationship between their levels. This notion, together with that of a *connected component* (2.14), will be the main technical tools employed in the analysis of the fine structure of ARS-fans carried out in this paper.

Proposition 5.11 in [DP5a] shows that the levels of an ARS-fan have a canonical structure of AOS-fans (1.1 (a)), that is, of fans in the category of abstract order spaces (cf. [M], § 3.1, pp. 37 ff.). We prove (Proposition 1.2) that inclusion of ideals induces AOS-morphisms between the corresponding levels (cf. 1.1 (c)). As an application we prove (Corollary 1.10) that the cardinality of a finite RS-fan, F , and that of its character space, X_F , are related by the identity $\text{card}(F) = 2 \cdot \text{card}(X_F) + 1$, an analog for RSs of a result known to hold for reduced special groups. \square

Preliminaries and Notation 1.1 (a) Given a real semigroup G and a saturated prime ideal I of G , we denote by G_I the RSG $(G/I) \setminus \{\pi(0)\}$, cf. [DP5a], Prop. 5.11. The congruence of G determined by the set of characters $\mathcal{H}_I = \{h \in X_G \mid Z(h) = I\}$ is denoted by \sim_I , cf. [DP5a], § 5.C. Every character $h \in \mathcal{H}_I$ induces a map $\hat{h} : G_I \rightarrow \{\pm 1\}$ defined by $\hat{h} \circ \pi_I = h$. The correspondence $h \mapsto \hat{h}$ is a bijection between the set $L_I(G) = \mathcal{H}_I$ and the space of orders X_{G_I} of G_I . (L_I stands for “ I -th level”; see item (b.ii) below). Thus, we can identify the set $L_I(G) \subseteq X_G$ with the AOS (X_{G_I}, G_I) . We shall systematically use this identification in the sequel, and unambiguously refer to the AOS structure of the set $L_I(G)$. In case G is a RS-fan, [DP5a], Prop. 5.11, shows that $L_I(G)$ is an AOS-fan.

(b) Let F be a RS-fan.

(i) We denote by $\text{Spec}(F)$ the set of all proper ideals of F .

(ii) For $I \in \text{Spec}(F)$ the set $L_I(F) = \{h \in X_F \mid Z(h) = I\}$ is called the **I -th level** of X_F .

(iii) For $f \in X_F$, the **depth** of f , denoted $d(f)$, is the order type of the set $\{g \in X_F \mid f \rightsquigarrow g\}$ under the order of specialization. (Since (X_F, \rightsquigarrow) is a root-system, the order \rightsquigarrow is total on this set.)

(iv) For $I \in \text{Spec}(F)$, the order type of the set $\{J \in \text{Spec}(F) \mid J \supseteq I\}$ under the (total) order of inclusion will be called the **depth** of I , denoted $d(I)$.

(v) The **length** of X_F , denoted $\ell(X_F)$, is the order type of the (totally ordered) set $\text{Spec}(F)$.

(c) (AOS- and ARS-morphisms; [M], § 2, pp. 23-24, and § 6, p. 103)

(i) Let $(X, G), (Y, H)$ be ARS's. A map $F : X \rightarrow Y$ is an **ARS-morphism** iff for all $a \in H$ there is $b \in G$ so that $\hat{a} \circ F = \hat{b}$. Here, for $x \in G$, $\hat{x} : X \rightarrow \mathbf{3}$ denotes the map “evaluation at x ”: $\hat{x}(\sigma) := \sigma(x)$, for $\sigma \in X$, and similarly for H .

(ii) The definition of an **AOS-morphism** is similar, with $(X, G), (Y, H)$ AOS's, and the evaluation maps taking values in $\{\pm 1\}$.

(d) If $f : G \rightarrow H$ is a RS-morphism (resp. RSG-morphism), the dual map $f^* : X_H \rightarrow X_G$ defined by $f^*(\gamma) := \gamma \circ f$ for $\gamma \in X_H$, is an ARS-morphism (resp., AOS-morphism). \square

Remarks. (a) Clearly, the union and the intersection of an inclusion chain of (proper) prime ideals in any ternary semigroup is a (proper) prime ideal. In particular, if F is a fan, the totally ordered set $(\text{Spec}(F), \subseteq)$ is (Dedekind) **complete**. \square

Proposition 1.2 *Let F be a RS-fan and let $I \subseteq J$ be ideals of F . With notation as in 1.1,*

(1) *The rule $a/J \mapsto a/I$ ($a \in F \setminus J$) defines a homomorphism of special groups $\iota_{JI} : F_J \rightarrow F_I$.*

(2) *The map $\kappa_{IJ} : L_I(F) \rightarrow L_J(F)$ assigning to each $g \in L_I(F)$ the unique element $h \in L_J(F)$ such that $g \rightsquigarrow h$ is an AOS-morphism.*

Proof. (1) (a) ι_{JI} is well-defined.

We must show: $a, b \in F \setminus J \wedge a \sim_J b \Rightarrow a \sim_I b$. Since $I \subseteq J$, this is clear from Lemma 5.10 of [DP5a] which states that, for an ideal K of F and $a, b \in F \setminus K$, $a \sim_J b \Leftrightarrow \exists z \notin K (az = bz)$.

Clearly, we have:

(b) ι_{JI} is a group homomorphism sending $-1/J$ to $-1/I$.

Since F_J is a RSG-fan, ι_{JI} is automatically a homomorphism of special groups.

(2) By (1) and 1.1 (d), the map $\iota_{JI}^* : X_{F_I} \rightarrow X_{F_J}$ dual to ι_{JI} is an AOS-morphism. The map κ_{IJ} is $\kappa_{IJ} = (\varphi_J)^{-1} \circ \iota_{JI}^* \circ \varphi_I$, where φ_I denotes the bijection $g \mapsto \widehat{g}$ ($g \in L_I(F)$), identifying $L_I(F)$ with X_{F_I} (1.1 (a)), and similarly for $L_J(F)$. It only remains to prove $g \rightsquigarrow \kappa_{IJ}(g)$, for $g \in L_I(F)$. To ease notation, write $h = \kappa_{IJ}(g)$. According to Lemma 0.1 (4) we must show $Z(g) \subseteq Z(h)$ and $a \notin Z(h) \Rightarrow g(a) = h(a)$. The inclusion of zero-sets is $I \subseteq J$. Let $a \notin Z(h) = J$. Since:

$\varphi_J(h) = \varphi_J(\kappa_{IJ}(g)) = \iota_{JI}^*(\varphi_I(g)) = \varphi_I(g) \circ \iota_{JI}$, $\varphi_I(g)(a/I) = g(a)$ and $\varphi_J(h)(a/J) = h(a)$,
(cf. 1.1), we get,

$$h(a) = (\varphi_I(g))(\iota_{JI}(a/J)) = \varphi_I(g)(a/I) = g(a),$$

as required. \square

Next we prove that the depth of an ideal in a fan is the same as the depth of any element in the corresponding level; in particular, elements of the same depth belong to the same level.

Proposition 1.3 *Let F be a RS-fan. For $f \in X_F$ we have $d(f) = d(Z(f))$; equivalently, the sets $\{g \in X_F \mid f \rightsquigarrow g\}$ (ordered under specialization) and $\{J \in \text{Spec}(F) \mid J \supseteq Z(f)\}$ (ordered under inclusion) are order-isomorphic.*

Proof. To ease notation, set $f \uparrow = \{g \in X_F \mid f \rightsquigarrow g\}$ and $I \uparrow = \{J \in \text{Spec}(F) \mid J \supseteq I\}$ ($I \in \text{Spec}(F)$). The required order isomorphism is the map $Z : f \uparrow \rightarrow Z(f) \uparrow$ assigning to each $g \in f \uparrow$ its zero-set. That

(a) Z is increasing, and (b) Z is surjective,

is clear, from $g \rightsquigarrow h \Rightarrow Z(g) \subseteq Z(h)$ and Proposition 0.3 (2), respectively. That

(c) Z is injective.

follows from 0.2 (3). \square

A trivial variant of the proof of 1.3 gives:

Proposition 1.4 *Let F be a RS-fan. Given $f_1, f_2 \in X_F$ such that $f_1 \rightsquigarrow f_2$, the intervals $\{g \in X_F \mid f_1 \rightsquigarrow g \rightsquigarrow f_2\}$ (ordered under specialization) and $\{J \in \text{Spec}(F) \mid Z(f_1) \subseteq J \subseteq Z(f_2)\}$ (ordered under inclusion) are order-isomorphic.* \square

The results in the next two Lemmas will be frequently used in this and in subsequent sections.

Lemma 1.5 *Let G be a RS and let $g_1, \dots, g_r, h \in X_G$ be so that $\bigcup_{i=1}^r Z(g_i) \subseteq Z(h)$. For $i = 1, \dots, r$, let $f_i \in X_G$ be such that $g_i \rightsquigarrow f_i$ and $Z(g_i) \subseteq Z(f_i) \subseteq Z(h)$. Then,*

$$(*) \quad h \cdot g_1 \cdot \dots \cdot g_r = h \cdot f_1 \cdot \dots \cdot f_r.$$

Note. The products in (*) may not be in X_G .

Proof. Obviously, (*) holds whenever $x \in Z(h)$. If $x \notin Z(h)$, from the assumptions we get $x \notin \bigcup_{i=1}^r Z(g_i)$ and $x \notin \bigcup_{i=1}^r Z(f_i)$. Since $g_i \rightsquigarrow f_i$, we get $g_i(x) = f_i(x)$ for $i = 1, \dots, r$ (Lemma 0.1 (4)), and (*) follows. \square

Lemma 1.6 *Let F be a RS-fan. Then,*

(a) *For $i = 1, \dots, r$, with r odd, let $g_i, h_i \in X_F$ be such that $g_i \rightsquigarrow h_i$. Then, $g_1 \cdot \dots \cdot g_r \rightsquigarrow h_1 \cdot \dots \cdot h_r$.*

(b) *Let $h_1, h_2, f, g, k \in X_F$ be such that $f, g \rightsquigarrow h_1$, $k \rightsquigarrow h_2$, and $Z(h_1) \subseteq Z(h_2)$. Then, $f g k \rightsquigarrow h_2$.*

Note. Here the products are in X_F as the number of factors is odd.

Proof. (a) For $i = 1, \dots, r$ we have $h_i^2 = h_i g_i$ (Lemma 0.1 (5)). Multiplying these equalities termwise gives $(h_1 \cdot \dots \cdot h_r)^2 = (h_1 \cdot \dots \cdot h_r)(g_1 \cdot \dots \cdot g_r)$, which proves the assertion.

(b) By Lemma 0.1 we must prove $h_2^2 = h_2(fgk)$. Obviously, this equality holds at every $x \in Z(h_2)$. If $x \notin Z(h_2)$, then $x \notin Z(h_1)$, and $f, g \rightsquigarrow h_1$ implies $h_1(x) = f(x) = g(x) \neq 0$; also $k \rightsquigarrow h_2$ implies $h_2(x) = k(x) \neq 0$, whence $f(x)g(x) = 1$ and $h_2(x)k(x) = 1$. This yields $(h_2fgk)(x) = (f(x)g(x))(h_2(x)k(x)) = 1$. On the other hand, $(h_2(x))^2 = 1$, proving that the required identity holds at $x \notin Z(h_2)$ as well. \square

Our last result in this section, Corollary 1.10, shows that if F is a finite RS-fan and X_F its character space, then $\text{card}(F) = 2 \cdot \text{card}(X_F) + 1$. This identity is the analog of a well-known result relating the cardinalities of a finite RSG-fan and its space of orders ([ABR], p. 75). The result follows from a more general observation, valid for RS-fans of arbitrary cardinality.

Proposition 1.7 *Let $I \subset J$ be consecutive ideals of a RS-fan F (with, possibly, $J = F$). Then,*

(i) *Under product induced by F , $J \setminus I$ is a group of exponent 2 with unit x^2 for any $x \in J \setminus I$ (and distinguished element $-1 = -x^2$).*

(ii) *The restriction of the quotient map $\pi_I \upharpoonright (J \setminus I) : J \setminus I \longrightarrow F_I = F/I \setminus \{\pi_I(0)\}$ is a group isomorphism preseving the distinguished element -1 .*

Proof. (i) Since I is prime, $J \setminus I$ is closed under product. Given $x, y \in J \setminus I$, we must prove $x^2 = y^2$ (which implies $x^2y = y^3 = y$). By the separation theorem for TSs ([DP1], Thm. 1.9, pp. 103-104) it suffices to show that $h(x^2) = h(y^2)$ for all $h \in X_F$. If $J \subseteq Z(h)$, then $h(x^2) = h(y^2) = 0$. If $Z(h) \subseteq I$, then $h(x), h(y) \neq 0$, whence $h(x^2) = h(y^2) = 1$.

(ii) Clearly, $\pi_I(x) \neq \pi_I(0)$, i.e., $\pi_I(x) \in F_I$, for all $x \in J \setminus I$, and π_I preserves product.

— $\pi_I \upharpoonright (J \setminus I)$ is injective.

Suppose $\pi_I(x) = \pi_I(y)$, i.e., $x \sim_I y$, with $x, y \in J \setminus I$. By [DP5a], Lemma 5.10 (cf. proof of 1.2), $xz = yz$ for some $z \notin I$. To prove $x = y$, let $h \in X_F$. If $J \subseteq Z(h)$, then $h(x) = h(y) = 0$. If $Z(h) \subseteq I$, then $h(z) \neq 0$, and we get $h(x) = h(y)$.

— $\pi_I(x^2) = \pi_I(1)$, for $x \in J \setminus I$.

Clear, for $Z(h) = I$ implies $h(x^2) = 1$. In particular, π_I preserves -1 .

— $\pi_I \upharpoonright (J \setminus I)$ is onto F_I .

Let $p \in F_I$; then, $p = \pi_I(q)$ with $q \notin I$. Taking $z \in J \setminus I$, we have $qz^2 \in J \setminus I$, whence $\pi_I(qz^2) = \pi_I(q)\pi_I(z^2) = \pi_I(q)\pi_I(1) = \pi_I(q) = p$. \square

Notation 1.8 Let F be a finite RS-fan, and let

$$\{0\} = I_n \subset I_{n-1} \subset \dots \subset I_2 \subset I_1 \subset F = I_0$$

be the set of all its ideals; thus, for $1 \leq d \leq n$, I_d is the ideal of depth d . We set $F_d = F_{I_d} = (F/I_d) \setminus \{\pi_d(0)\}$, where $\pi_d : F \longrightarrow F/I_d$ denotes the canonical quotient map. We also write L_d for L_{I_d} ; cf. 1.1 (b). \square

Clearly, $F \setminus \{0\} = \bigcup_{d=1}^n (I_{d-1} \setminus I_d)$ (disjoint union), whence, by 1.7 we have $\text{card}(F) = \sum_{d=1}^n \text{card}(I_{d-1} \setminus I_d) + 1 = \sum_{d=1}^n \text{card}(F_d) + 1$. Since the levels partition X_F , 1.1 yields:

Proposition 1.9 *For any finite RS-fan F , $\text{card}(X_F) = \sum_{d=1}^n \text{card}(L_d) = \sum_{d=1}^n \text{card}(X_{F_d})$.* \square

Corollary 1.10 *For a finite RS-fan, F , $\text{card}(F) = 2 \cdot \text{card}(X_F) + 1$.*

Proof. Since the F_d are finite RSG-fans ([DP5a], Prop. 5.11), we know that $\text{card}(F_d) = 2 \cdot \text{card}(X_{F_d})$ for $1 \leq d \leq n$ (see [ABR], p. 75). The result follows, then, from Proposition 1.7 and the preceding cardinality identities. \square

2 Involutions of ARS-fans

Notation 2.1 In addition to the notation introduced in Definition 1.1, for $J \subseteq I$ in $\text{Spec}(F)$ we define the sets:

$$S_J^I = \{h \in L_I \mid \exists g \in X_F (g \rightsquigarrow h \wedge Z(g) = J)\}.$$

$$C_J^I = \{h \in S_J^I \mid \forall g' \in X_F (g' \rightsquigarrow h \Rightarrow J \subseteq Z(g'))\}.$$

That is, S_J^I consists of those elements of level I having predecessors of level J or lower in the specialization partial order; C_J^I is the set of elements in L_I having predecessors at level J but not lower. \square

Remarks 2.2 (i) For $I \in \text{Spec}(F)$, $S_{\{0\}}^I = C_{\{0\}}^I$, and $S_I^I = L_I$. (Recall that $\{0\}$ is the smallest member of $\text{Spec}(F)$, i.e., the zero-set of the lowest level of X_F .)

(ii) For $J \subseteq I$ in $\text{Spec}(F)$, $S_J^I \neq \emptyset$.

Proof. Let $g \in X_F$ be such that $Z(g) = J$ (exists by Proposition 0.3 (3)). If h is the unique \rightsquigarrow -successor of g of level I (Proposition 0.3 (2)), then $h \in S_J^I$.

(iii) For $J \subseteq I$ in $\text{Spec}(F)$, $S_J^I = \text{Im}(\kappa_{JI})$, where $\kappa_{JI} : L_J(F) \rightarrow L_I(F)$ is the AOS-morphism defined in Proposition 1.2 (2).

(iv) For $J \subseteq I$ in $\text{Spec}(F)$, $S_J^I \supseteq \bigcup \{C_{J'}^I \mid J' \in \text{Spec}(F) \text{ and } J' \subseteq J\}$. (Note that $C_{J'}^I$ may be empty for some $J' \subseteq J$.)

(v) For $J \subseteq I$ in $\text{Spec}(F)$, $C_J^I = S_J^I \setminus \bigcup \{S_{J'}^I \mid J' \in \text{Spec}(F) \text{ and } J' \subset J\}$.

(vi) For $J, J' \subseteq I$ in $\text{Spec}(F)$, $J \neq J'$, we have $C_J^I \cap C_{J'}^I = \emptyset$. \square

In order to render later arguments as transparent as possible, we recall the following simple (and well-known) facts about fans in the categories **RSG** and **AOS**.

Lemma 2.3 Let $g : H \rightarrow G$ be a SG-homomorphism between RSG-fans, and let $g^* : (X_G, G) \rightarrow (X_H, H)$ denote the AOS-morphism dual to g (cf. 1.1 (d)). Then,

- (1) With representation induced by that of H , $\text{Im}(g)$ is a RSG-fan, and G is isomorphic to the extension of $\text{Im}(g)$ by the exponent-two group $\Delta = G/\text{Im}(g)$.
- (2) $(\text{Im}(g^*), H/\ker(g))$ is an AOS-fan.

Remarks 2.4 (a) For the definition of extension of a SG by a group of exponent two, see [DM1], Ex. 1.10, p. 10.

(b) By the duality between RSGs and AOSs ([DM1], Ch. 3), the dual statement holds as well: given an AOS-morphism of (AOS-)fans, $\kappa : (X, G) \rightarrow (Y, H)$, the assertions (1) and (2) hold with $g := \kappa^*$ (the SG-morphism dual to κ), and with $g^* = \kappa$. \square

Sketch of proof of 2.3. (1) The first assertion is easily checked. For the second, $\text{Im}(g)$ is a direct summand of the group G . Let $pr : G \rightarrow \text{Im}(g)$ be the projection onto the factor $\text{Im}(g)$; pr is a SG-morphism (G and $\text{Im}(g)$ are fans), and is the identity on $\text{Im}(g)$. The isomorphism between G and $\text{Im}(g)[\Delta]$ is $f(a) = \langle pr(a), a/\text{Im}(g) \rangle$, for $a \in G$.

(2) Recall that g^* is defined by composition, $g^*(\sigma) = \sigma \circ g$ ($\sigma \in X_G$), see 1.1 (d), and that $\text{Im}(g^*)^\perp = \bigcap \{\ker(\gamma) \mid \gamma \in \text{Im}(g^*)\} = \bigcap \{\ker(\sigma \circ g) \mid \sigma \in X_G\}$. Routine checking from these definitions proves that $\text{Im}(g^*)$ is closed under product of any three members (since X_G is), and that $\text{Im}(g^*)^\perp = \ker(g)$ (since $\bigcap \{\ker(\sigma) \mid \sigma \in X_G\} = \{1\}$), whence $\text{Im}(g^*) \subseteq X_{H/\ker(g)}$.

Clearly, the map $\bar{g} : H/\ker(g) \rightarrow \text{Im}(g)$ induced by g is an SG-isomorphism. Thus, we have a commutative diagram of SG-morphisms:

$$\begin{array}{ccccc}
H & \xrightarrow{g} & \text{Im}(g) & \xrightleftharpoons[\text{pr}]{f} & G \stackrel{f}{\cong} \text{Im}(g)[\Delta] \\
& \searrow \pi & \uparrow \bar{g} & & \\
& & H/\ker(g) & &
\end{array}$$

It only remains to show that $\text{Im}(g^*) \supseteq X_{H/\ker(g)}$. Any SG-character $\gamma : H/\ker(g) \rightarrow \mathbb{Z}_2$ can be lifted to a map $\sigma : G \rightarrow \mathbb{Z}_2$, via the identification of G with $\text{Im}(g)[\Delta]$, as follows: for each $a \in G$ there is $b \in H$ such that $\text{pr}(a) = g(b)$. We set $\sigma(a) = \gamma(b/\ker(g)) = \gamma(\pi(b))$. In terms of the diagram above, we have: $\sigma = \gamma \circ (\bar{g})^{-1} \circ \text{pr}$. It follows that σ is a well-defined SG-morphism, i.e., $\sigma \in X_G$, and (since $\text{pr} \circ g = g$ and $(\bar{g})^{-1} \circ g = \pi$), $g^*(\sigma) = \sigma \circ g = \gamma \circ \pi$. \square

Lemma 2.3, together with 2.2 (iii) and 1.2 (2), gives:

Corollary 2.5 *Let F be a RS-fan, and let $J \subseteq I$ be in $\text{Spec}(F)$. The set S_J^I is an AOS-fan. Indeed, it is a sub-fan of $L_I(F)$, when the latter is endowed with its structure of AOS-fan, as indicated in 1.1. More generally, if $\mathcal{F} \subseteq L_J(F)$ is an AOS-fan, the set $S_J^I(\mathcal{F}) = \{h \in L_I \mid \exists g \in \mathcal{F} (g \rightsquigarrow h)\}$ is an AOS-subfan of $L_I(F)$.*

Proof. The first assertion is a special case of the second (with $\mathcal{F} = L_J(F)$). For the latter, observe that $S_J^I(\mathcal{F}) = \kappa_{JI}[\mathcal{F}] = \text{Im}(\kappa_{JI}[\mathcal{F}])$ and use Remark 2.4 (b). \square

The following definition will have a crucial role in the sequel:

Definition 2.6 Let F be a RS-fan, let $g_1, g_2 \in X_F$, and fix $I \in \text{Spec}(F)$ so that $Z(g_1), Z(g_2) \subseteq I$. We define a map $\varphi_I^{g_1, g_2} : L_I(F) \rightarrow L_I(F)$ as follows: for $h \in L_I(F)$,

$$\varphi_I^{g_1, g_2}(h) = h g_1 g_2. \quad \square$$

Note. Since $Z(g_i) \subseteq I = Z(h)$ ($i = 1, 2$), we have $Z(h g_1 g_2) = I$, whence $h g_1 g_2 \in L_I$.

Fact 2.7 *With notation as in Definition 2.6, let $J \in \text{Spec}(F)$ be such that $Z(g_1) \cup Z(g_2) \subseteq J \subseteq I$, and for $i = 1, 2$, let g'_i be the unique \rightsquigarrow -successor of g_i of level J . Then, $\varphi_I^{g_1, g_2} = \varphi_I^{g'_1, g'_2}$. Thus, in 2.6 we may assume $Z(g_1) = Z(g_2)$.*

Proof. Lemma 1.5 shows that $h g_1 g_2 = h g'_1 g'_2$, for $h \in L_I$. \square

Theorem 2.8 *With notation as in Definition 2.6, we have:*

- (a) $\varphi_I^{g_1, g_2}$ is an AOS-automorphism of L_I .
- (b) $\varphi_I^{g_1, g_2}$ is an involution: for $h \in L_I$, $\varphi_I^{g_1, g_2}(\varphi_I^{g_1, g_2}(h)) = h$.
- (c) For $i = 1, 2$, let h_i be the unique \rightsquigarrow -successor of g_i in L_I . Then, $\varphi_I^{g_1, g_2}(h_1) = h_2$.

In particular,

- (d) If g_1, g_2 have a common specialization h at some level $I \supseteq Z(g_1), Z(g_2)$, then h is a fixed point of $\varphi_I^{g_1, g_2}$.
- (e) Let $J \subseteq I$ be in $\text{Spec}(F)$. Assume $Z(g_1), Z(g_2) \subseteq J$, and let $h_1 \in L_J$, $h_2 \in L_I$. Then,

$$h_1 \rightsquigarrow h_2 \Rightarrow \varphi_J^{g_1, g_2}(h_1) \rightsquigarrow \varphi_I^{g_1, g_2}(h_2).$$

For the proof of this Theorem we will need an improvement on 1.1 (a), valid for fans but not for arbitrary RSs; namely:

Fact 2.9 *Let F be a RS-fan, and I be an ideal of F . Any $g \in X_F$ such that $\underline{Z(g)} \subseteq I$ induces a SG-character $\hat{g} : F_I \rightarrow \mathbb{Z}_2$, by setting $\hat{g} \circ \pi_I = g$.*

Proof. The only delicate point is well-definedness: for $a \in F \setminus I$, $a \sim_I 1 \Rightarrow g(a) = 1$. By [DP5a], Lemma 5.10, $a \sim_I 1$ means $az = z$ for some $z \notin I$ (see proof of 1.2); then $g(z) \neq 0$, and taking images under g in this equality yields $g(a) = 1$. \square

Proof of Theorem 2.8. We begin by proving:

(b) For $h \in L_I(F)$, $\varphi_I^{g_1, g_2}(\varphi_I^{g_1, g_2}(h)) = h g_1^2 g_2^2$. But $h g_1^2 g_2^2 = h$; this is clear if $h(x) = 0$ ($x \in F$); if $h(x) \neq 0$, then $g_i(x) \neq 0$ (since $Z(g_i) \subseteq Z(h)$), and hence $g_i^2(x) = 1$ ($i = 1, 2$), proving the stated identity, and item (b).

(a) i) $\varphi_I^{g_1, g_2}$ is an AOS-morphism.

Since F_I is the RSG-fan dual to $L_I(F)$, we must show (see 1.1 (c)):

(*) For every $\alpha \in F_I$ there is $\beta \in F_I$ such that $\hat{\alpha} \circ \varphi_I^{g_1, g_2} = \hat{\beta}$,

where $\hat{\alpha} : X_{F_I} \rightarrow \mathbb{Z}_2$ denotes evaluation at α . We claim that $\beta = \alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha)$ does the job. By Fact 2.9, $\hat{g}_i(\alpha) \in \mathbb{Z}_2$ ($i = 1, 2$), whence $\beta \in F_I$. For $h \in L_I(F)$ we have:

$$(\hat{\alpha} \circ \varphi_I^{g_1, g_2})(h) = \hat{\alpha}(h g_1 g_2) = h(\alpha) \hat{g}_1(\alpha) \hat{g}_2(\alpha) = h(\alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha)) = h(\beta) = \hat{\beta}(h),$$

as required. Note that (b) implies

ii) $\varphi_I^{g_1, g_2}$ is bijective.

iii) The dual map $(\varphi_I^{g_1, g_2})^* : F_I \rightarrow F_I$ is also bijective.

Item (i) proves that, for $\alpha \in F_I$, $(\varphi_I^{g_1, g_2})^*(\alpha) = \alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha)$. For injectivity, assume $\alpha \hat{g}_1(\alpha) \hat{g}_2(\alpha) = 1$; if $\hat{g}_1(\alpha) \hat{g}_2(\alpha) = -1$, then $\alpha = -1$, whence (as \hat{g}_i is a SG-character), $\hat{g}_i(\alpha) = -1$ ($i = 1, 2$), and $\alpha = 1$, contradiction. Thus, $\hat{g}_1(\alpha) \hat{g}_2(\alpha) = 1$, which entails $\alpha = 1$. For surjectivity, given $\beta \in F_I$, set $\alpha = \beta \hat{g}_1(\beta) \hat{g}_2(\beta)$. Then, $\hat{g}_1(\alpha) = \hat{g}_2(\beta)$ and $\hat{g}_2(\alpha) = \hat{g}_1(\beta)$, whence $(\varphi_I^{g_1, g_2})^*(\alpha) = \beta$.

(c) We must prove $h_1 g_1 g_2 = h_2$. This clearly holds at any $x \in Z(h_1) = Z(h_2)$. If $x \notin Z(h_i)$ ($i = 1, 2$), then $x \notin Z(g_i)$; since $g_i \rightsquigarrow h_i$, it follows $h_i(x) = g_i(x) \neq 0$ (Lemma 0.1 (4)), and $h_i(x) g_i(x) = 1$; hence, $h_1 g_1 g_2(x) = g_2(x) = h_2(x)$.

(e) Lemma 1.6 (a) immediately implies $h_2^2 = h_2 h_1 \Rightarrow (h_2 g_1 g_2)^2 = (h_2 g_1 g_2)(h_1 g_1 g_2)$. \square

By use of these involutions we obtain a number of regularity results concerning the order structure of ARS-fans.

Proposition 2.10 *Let F be a RS-fan. For $J \subseteq J_1 \subseteq J_2 \subseteq I$ in $\text{Spec}(F)$, and $h \in S_J^I$ set:*

$$B^{J_1, J_2}(h) = \{g \in S_{J_1}^{J_2} \mid g \rightsquigarrow h\}, \quad \text{and} \quad A^{J_1, J_2}(h) = \{g \in C_{J_1}^{J_2} \mid g \rightsquigarrow h\}.$$

Then,

(a) *For $h_1, h_2 \in S_J^I$, we have $\text{card}(B^{J_1, J_2}(h_1)) = \text{card}(B^{J_1, J_2}(h_2))$.*

(b) *For $h_1, h_2 \in C_J^I$, we have $\text{card}(A^{J_1, J_2}(h_1)) = \text{card}(A^{J_1, J_2}(h_2))$.*

Remark. The assumptions of the Proposition guarantee that the sets $B^{J_1, J_2}(h)$ are non-empty. In fact, given $h \in S_J^I$, there is $u \rightsquigarrow h$ so that $Z(u) \subseteq J$; set $J' = Z(u)$. Since $J' \subseteq J \subseteq J_2$, u has a unique \rightsquigarrow -successor g in L_{J_2} . But $u \rightsquigarrow g$, h and $J_2 = Z(g) \subseteq I = Z(h)$ imply $g \rightsquigarrow h$ (Lemma 0.2 (3)). Since $J' \subseteq J \subseteq J_1$, we conclude that $g \in S_{J_1}^{J_2}$, i.e., $g \in B^{J_1, J_2}(h)$.

The sets $A^{J_1, J_2}(h)$ may be empty for some choices of h and the J_i 's. However, if $h \in C_J^I$ and $J_1 = J$, we have $A^{J_1, J_2}(h) \neq \emptyset$. Indeed, in this case the element $g \in S_J^{J_2}$ constructed above is in $C_J^{J_2}$, for if $g \in S_{J'}^{J_2}$ for some $J' \subset J$, then $g \rightsquigarrow h$ would imply $h \in S_{J'}^I$, contrary to the assumption $h \in C_J^I$. \square

Proof of Proposition 2.10. (a) With J_1, J_2 as in the statement, write B_i for $B^{J_1, J_2}(h_i)$ ($i =$

1, 2). The assumption $h_i \in S_J^I$ implies the existence of elements $u_i \in X_F$ so that $u_i \rightsquigarrow h_i$ and $Z(u_i) \subseteq J$. Replacing u_i by its unique successor of level J we may assume $Z(u_i) = J$ (see 2.7). We fix u_i 's with these properties throughout the proof, and for $J \subseteq J' \subseteq I$ we denote by $\varphi_{J'}$ the involution $\varphi_{J'}^{u_1, u_2}$ of $L_{J'}$ defined in 2.6.

Since the maps $\varphi_{J'}$ are bijective, it is enough to prove $\varphi_{J_2}[B_1] = B_2$. Further, since φ_{J_2} is an involution it suffices just to prove the inclusion \subseteq , i.e.,

$$(*) \quad g \in S_{J_1}^{J_2} \text{ and } g \rightsquigarrow h_1 \Rightarrow \varphi_{J_2}(g) \rightsquigarrow h_2 \text{ and } \varphi_{J_2}(g) \in S_{J_1}^{J_2}.$$

$$(i) \quad \varphi_{J_2}(g) = g u_1 u_2 \rightsquigarrow h_2.$$

Immediate consequence of Lemma 1.6 (b), since $g, u_1 \rightsquigarrow h_1$ and $u_2 \rightsquigarrow h_2$.

$$(ii) \quad \varphi_{J_2}(g) \in S_{J_1}^{J_2}.$$

Since $g \in S_{J_1}^{J_2}$, there is $v \rightsquigarrow g$ so that $Z(v) = J_1 \supseteq J = Z(u_i)$ ($i = 1, 2$); thus, v is in the domain of $\varphi_{Z(v)} = \varphi_{Z(v)}^{u_1, u_2}$, and Theorem 2.8 (e) gives $\varphi_{Z(v)}(v) \rightsquigarrow \varphi_{J_2}(g)$, proving (ii) and item (a).

(b) Write A_i for $A^{J_1, J_2}(h_i)$ ($i = 1, 2$). As above, it suffices to prove the analogue of (*):

$$(**) \quad g \in C_{J_1}^{J_2} \text{ and } g \rightsquigarrow h_1 \Rightarrow \varphi_{J_2}(g) \rightsquigarrow h_2 \text{ and } \varphi_{J_2}(g) \in C_{J_1}^{J_2},$$

where h_1, h_2 are now assumed to be in C_J^I . In fact, by (*) it only remains to show:

$$(iii) \quad \text{There is no } w \in X_F \text{ such that } Z(w) \subset J_1 \text{ and } w \rightsquigarrow \varphi_{J_2}(g).$$

Otherwise, we would have $w \rightsquigarrow \varphi_{J_2}(g) \rightsquigarrow h_2$ (the last relation holding by (*)). Since $h_2 \in C_J^I$, we get $J \subseteq Z(w)$, and since $Z(u_i) = J$, $\varphi_{Z(w)}(w)$ is defined. Theorem 2.8 (e) applied to the first of the preceding inequalities yields: $\varphi_{Z(w)}(w) \rightsquigarrow \varphi_{J_2}(\varphi_{J_2}(g)) = g$. This, together with $\varphi_{Z(w)}(w) \in L_{Z(w)}$ and $Z(w) \subset J_1$, contradicts the assumption $g \in C_{J_1}^{J_2}$, proving (iii), and item (b). \square

A slight variant of the argument proving Proposition 2.10 yields:

Proposition 2.11 *Let F be a RS-fan and let $J \subseteq I$ be in $\text{Spec}(F)$. For $g_1, g_2 \in X_F$ such that $Z(g_i) \subseteq J$ ($i = 1, 2$), the map $\varphi_J^{g_1, g_2}$ is a permutation of S_J^I and of C_J^I .* \square

For a RS-fan, F , and $h \in X_F$, we denote by $P_h = \{g \in X_F \mid g \rightsquigarrow h\}$ the root-system of predecessors of h under specialization. We begin by proving:

Proposition 2.12 (1) P_h is an ARS-fan. In particular,

(2) Any connected component of an ARS-fan is an ARS-fan.

Proof. (1) Lemma 0.1 (2) shows that $g \rightsquigarrow h$ iff $T = h^{-1}[1] \subseteq g^{-1}[1]$. With notation as in [M], § 6.3, p. 110, and § 6.6, p. 126, the latter condition just means $g \in U(T)$, i.e., P_h is the saturated set $U(T) (= W(T) \cap U(T^2))$. [M], Cor. 6.6.8, p. 126, proves that sets of this form are ARSs. Lemma 1.6 (a) shows that it is closed under products of three elements, hence a fan by the results of [DP5a], § 3.

(2) Follows from (1) by taking h to be the (unique) \rightsquigarrow -top element of the given connected component. \square

Continuing the analysis of (ARS-)fans of the form P_h , we show:

Theorem 2.13 *Let F be a RS-fan and let $J \subseteq I$ be in $\text{Spec}(F)$. Let $h_1 \in C_J^I$, $h_2 \in S_J^I$. For $i = 1, 2$, we write P_i for P_{h_i} . Then,*

(1) *There is an ARS-embedding φ of P_1 into P_2 . Further, $\varphi[P_1] = \{u \in P_2 \mid J \subseteq Z(u)\}$. In particular, φ is an order-embedding of (P_1, \rightsquigarrow) into (P_2, \rightsquigarrow) .*

(2) *If, in addition, $h_2 \in C_J^I$, then φ is an isomorphism of ARSs.*

Proof. Our assumption on the h_i 's guarantees the existence of $u_1, u_2 \in L_J$ so that $u_i \rightsquigarrow h_i$ ($i = 1, 2$). For $J \subseteq J' \subseteq I$ in $\text{Spec}(F)$ let $\varphi_{J'}$ denote the involution $\varphi_{J'}^{u_1, u_2}$ of $L_{J'}$ (Definition 2.6).

(1) We construct $\varphi : P_1 \longrightarrow P_2$ by “collecting together” all the relevant maps $\varphi_{J'}$ ($J \subseteq J' \subseteq I$): given $g \in L_{J'}$, $g \rightsquigarrow h_1$, we set

$$\varphi(g) = \varphi_{J'}(g).$$

Since the levels $L_{J'}$ are pairwise disjoint, φ is well-defined.

i) $\varphi[P_1] \subseteq P_2$.

By Theorem 2.8 (e), $g \rightsquigarrow h_1$ implies $\varphi_{J'}(g) \rightsquigarrow \varphi_I(h_1)$. Since h_i is the unique successor of u_i at level I , 2.8 (c) yields $\varphi_I(h_1) = h_2$, whence $\varphi_{J'}(g) \rightsquigarrow h_2$, as required. Note this also gives $J \subseteq J' = Z(\varphi(g))$.

ii) $\{u \in P_2 \mid J \subseteq Z(u)\} \subseteq \varphi[P_1]$.

Let u be in the left-hand side, with $J' = Z(u)$, say. Set $v = \varphi_{J'}(u)$; then, $\varphi_{J'}(v) = u$ (2.8 (b)). By 1.6 (b), $u_1 \rightsquigarrow h_1$ and $u, u_2 \rightsquigarrow h_2$ imply $u u_1 u_2 = \varphi_{J'}(u) = v \rightsquigarrow h_1$, i.e., $v \in P_1$. Hence $\varphi(v) = u \in \varphi[P_1]$.

iii) φ is injective.

This is clear using 2.8 (b), since $Z(\varphi(g)) = Z(g)$ for $g \in P_1$.

iv) φ is an ARS-morphism.

The proof is similar to that of item (a) in Theorem 2.8. The statement to be proved is:

(†) For every $a \in F$ there is $b \in F$ such that $\widehat{a/T_2} \circ \varphi = \widehat{b/T_1}$,

where, for $i = 1, 2$, $T_i = h_i^{-1}[1]$, $P_i = U(T_i)$, $\widehat{a/T_2} : P_2 \longrightarrow \mathbf{3}$ is the evaluation map: $\widehat{a/T_2}(g) = \widehat{g}(a/T_2) = g(a)$, for $g \in P_2$, and similarly for $\widehat{b/T_1} : P_1 \longrightarrow \mathbf{3}$. (Note that $g \in P_2 = U(T_2)$ ensures that $\widehat{a/T_2}$ depends only on the congruence class of a modulo T_2 .)

The conclusion of (†) can equivalently be written as $\widehat{\varphi(g)}(a/T_2) = \widehat{g}(b/T_1)$, i.e., $(u_1 u_2 g)(a) = g(b)$. Since $u_i(a) \in \{0, 1, -1\}$ ($i = 1, 2$), it is clear that the element $b = a u_1(a) u_2(a) \in F$ verifies (†); see 2.8 (a).

(2) Since $h_2 \in C_J^I$, the preceding construction can be performed with the roles of h_1 and h_2 reversed. Routine verification using 2.8(b) shows that the map obtained is φ^{-1} , which then is an ARS-morphism, proving that φ is an ARS-isomorphism. \square

Proposition 2.10 and Theorem 2.13 provide significant information on the structure of the connected components of ARS-fans.

Definition and Remarks 2.14 (a) Let (X, \preceq) be a root-system and let $g_1, g_2 \in X$. Define:

$$g_1 \equiv_C g_2 \quad \text{iff} \quad g_1, g_2 \text{ have a common } \preceq\text{-upper bound.}$$

\equiv_C is an equivalence relation; its classes are called **connected components** of (X, \preceq) .

(b) The \rightsquigarrow -top elements of the connected components of an ARS-fan (X, F) have all the same level, namely the level determined by the maximal ideal M of F ; cf. Proposition 0.3 (3).

(c) Since every connected component of an ARS-fan is itself an ARS-fan, 2.12 (2), the zero-sets of its elements attain a lowest level, which can be explicitly determined, cf. Proposition 2.15 below. However, different components may have different lowest levels, see Corollary 2.17. \square

Notation. The sets L_I , S_J^I and C_J^I defined in 1.1 and 2.1 relativize in an obvious way to the connected components of a fan (X, F) ; if K is such a component and $J \subseteq I$ are in $\text{Spec}(F)$ we set:

$$L_I(K) = L_I \cap K, \quad S_J^I(K) = S_J^I \cap K, \quad \text{and} \quad C_J^I(K) = C_J^I \cap K.$$

Note that some (or all) of these sets may be empty, depending on I, J and the component K . Clearly, if h_0 is the \rightsquigarrow -top element of K , we have $L_I(K) = \{g \in L_I \mid g \rightsquigarrow h_0\}$, and similarly

for $S_J^I(K)$ and $C_J^I(K)$. $L_I(K) \neq \emptyset$ just means that K “reaches at least” the I -th level of X (possibly lower). \square

Proposition 2.15 *Let K be a connected component of an ARS-fan (X, F) . Let h_0 be the \rightsquigarrow -top element of K , and let $T = h_0^{-1}[1]$. Then, the lowest level of K (i.e., the smallest ideal I of F such that $L_I(K) \neq \emptyset$) is $I = \Gamma \cap -\Gamma$, where Γ is the saturated subsemigroup of F generated by $\text{Id}(F) \cdot T$.*

Note. The subsemigroup $\text{Id}(F) \cdot T$ may not be saturated, since $\text{Id}(F) \cdot T \cap -(\text{Id}(F) \cdot T)$ is not, in general, an ideal; see [DP5a], Cor. 3.10 (2).

Proof. With notation as in 2.12, we have $K = P_{h_0} = U(T) = \{g \in X \mid g \upharpoonright T = 1\} =$ the ARS $X_{F/T}$ (where $F/T = F/\sim_K$, with \sim_K denoting the congruence on F induced by K). Let $\pi_T : F \rightarrow F/T$ be the quotient map. The lowest level of $X_{F/T}$ is $\{0\}$; with K identified to a subset of X via the map $g \mapsto \widehat{g}$ ($\widehat{g} \circ \pi_T = g$), the corresponding ideal of F is $\pi_T^{-1}[0] = \{a \in F \mid a \sim_K 0\}$. Then, with the ideal I defined in the statement, we must prove, for $a \in F$:

$$a \sim_K 0 \iff a \in I.$$

(\Leftarrow) This follows from $I \subseteq Z(g)$ for all $g \in K$. Since $g \upharpoonright T = 1$, we get $\text{Id} \cdot T \subseteq P(g) = g^{-1}[0, 1]$; since $P(g)$ is a saturated subsemigroup, it comes $\Gamma \subseteq P(g)$. Hence, $x \in I = \Gamma \cap -\Gamma$ implies $g(x) = 0$.

(\Rightarrow) Assume $a \notin I$. In order to prove $a \not\sim_K 0$ we construct a character $g \in X$ such that $g \upharpoonright T = 1$ and $g(a) \neq 0$ (i.e., $g(a^2) = 1$). The ideal I is prime and saturated ([DP5a], 3.10 (1)). Since $I = \Gamma \cap -\Gamma$, there is a saturated subsemigroup S of F such that $\Gamma \subseteq S$ and S maximal with $S \cap -S = I$. By [DP1], Lemma 3.5, p. 114, $S \cup -S = F$, and S defines a character $g \in X$ with $Z(g) = I$, by setting $g \upharpoonright (S \setminus -S) = 1$, $g \upharpoonright (-S \setminus S) = -1$ and $g \upharpoonright I = 0$. Note that we have,

$$(\dagger) \quad I \cap a^2T = \emptyset.$$

Otherwise, there is $t \in T$ such that $a^2t \in I$; since I is prime and $a \notin I$, we get $t \in I$, contradicting $T \cap I = h_0^{-1}[1] \cap Z(h_0) = \emptyset$.

Since $a^2T \subseteq S$, (\dagger) implies $-S \cap a^2T = \emptyset$, whence $g \upharpoonright a^2T = 1$ by the definition of g . \square

Proposition 2.10 implies:

Corollary 2.16 *Let (X, F) be an ARS-fan and let K_1, K_2 be connected components of (X, F) . Then,*

- (1) *Let $I \in \text{Spec}(F)$; if $L_I(K_i) \neq \emptyset$ for $i = 1, 2$, then $\text{card}(L_I(K_1)) = \text{card}(L_I(K_2))$.*
- (2) *Let $J \subseteq J'$ be in $\text{Spec}(F)$, and assume $L_J(K_i) \neq \emptyset$ ($i = 1, 2$). Then, $\text{card}(S_J^{J'}(K_1)) = \text{card}(S_J^{J'}(K_2))$.*

Proof. (1) follows from (2), as $L_I = S_I^I$.

(2) Fix $i \in \{1, 2\}$. Let h_i be the \rightsquigarrow -top element of K_i . The assumption $L_J(K_i) \neq \emptyset$ implies that the sets $S_J^{J'}(K_i) = \{g \in S_J^{J'} \mid g \rightsquigarrow h_i\}$ are non-empty. Now, applying Proposition 2.10(a) with $I = M$ (= the maximal ideal of F), $J_1 = J$, $J_2 = J'$ we have $B^{J, J'}(h_i) = \{g \in S_J^{J'} \mid g \rightsquigarrow h_i\} = S_J^{J'}(K_i)$, and the result follows. \square

Remark. Assertion (2) of Corollary 2.16 fails, in general, if the sets $S_J^{J'}(K_i)$ are replaced by $C_J^{J'}(K_i)$, even if both sets $C_J^{J'}(K_i)$, $i = 1, 2$, are assumed non-empty. The snag is that $C_J^{J'}(K_i) \neq \emptyset$ does not imply that the \rightsquigarrow -top element h_i of K_i belongs to $C_J^M(K_i)$, a condition required for Proposition 2.10(b) to apply. It is easy to construct counterexamples. \square

Theorem 2.13 gives:

Corollary 2.17 *Let K_1, K_2 be connected components of the ARS-fan (X, F) . Let $I_1, I_2 \in \text{Spec}(F)$ be the lowest levels of K_1, K_2 , resp. (cf. 2.15). Then,*

- (1) *If $I_2 \subseteq I_1$, then K_1 endowed with the specialization order is (order-)isomorphic to the root-system obtained by deleting all levels $I \subset I_1$ in K_2 .*
- (2) *If $I_1 = I_2$, then K_1, K_2 are order-isomorphic.*

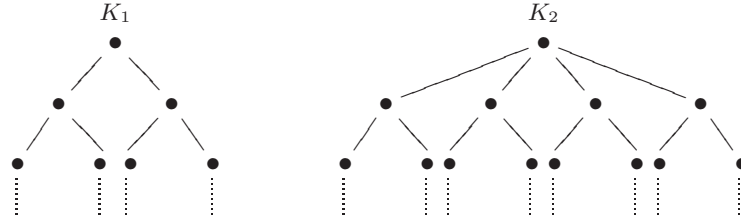
Proof. (1) Use Theorem 2.13 (1) with $I = M =$ the maximal ideal of F , $J = I_1$, and h_1, h_2 the \rightsquigarrow -top elements of K_1, K_2 , resp. The ARS-embedding $\varphi : K_1 \rightarrow K_2$ constructed therein verifies $\varphi[K_1] = \{u \in K_2 \mid I_1 \subseteq Z(u)\}$, which is exactly statement (1).

(2) follows from Theorem 2.13 (2). \square

2.18 Some impossible configurations.

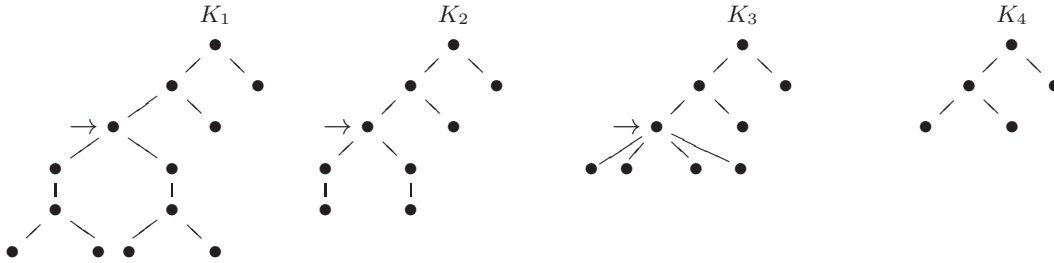
The preceding results show that there are strong constraints on the order structure of ARS-fans, especially when there is more than one connected component. We include a few examples to help the reader visualize the extent of those restrictions.

- (1) A configuration like



contradicts Corollary 2.16 (1).

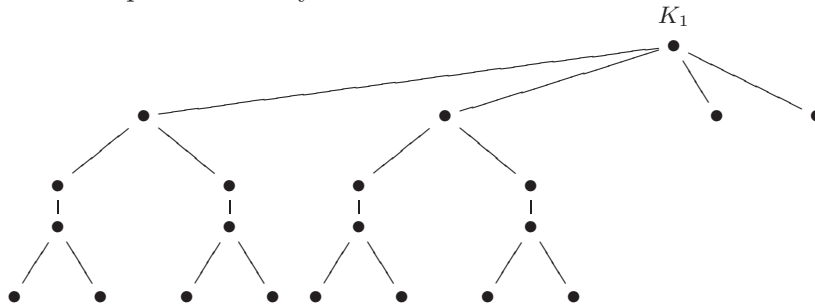
- (2) The four-component configuration

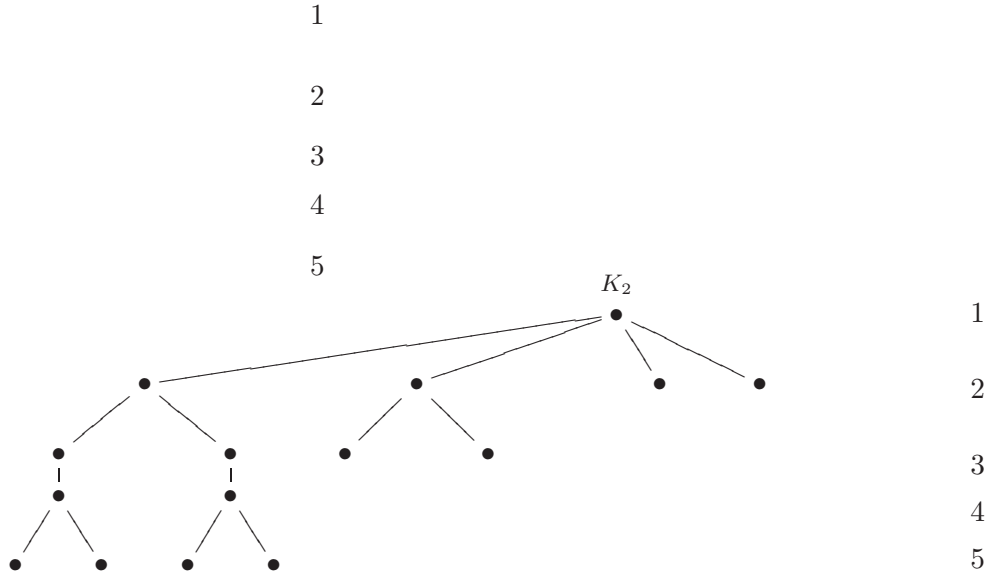


(where the components pairwise verify the conclusion of 2.16 (2)) is also impossible: $\text{card}(S_4^3) = 3$ is not a power of 2, and hence S_4^3 (shown with arrows) cannot be an AOS-fan (see Corollary 2.5). However, the same configuration with K_3 replaced by another copy of K_4 does not clash with either 2.16 or 2.17.

Note. Our notation here (and below) follows the convention introduced in 1.8 for finite fans. Thus, S_4^3 stands for the set $S_{I_4}^{I_3}$, see 2.1 and 3.1.

- (3) The two-component root-system





contradicts both Corollary 2.16 (2) ($\text{card}(S_4^3(K_1)) = 4$, but $\text{card}(S_4^3(K_2)) = 2$) and Corollary 2.17 (K_1 and K_2 have the same “length” but are not order-isomorphic). \square

3 The specialization root-system of finite ARS-fans

In this section we shall mostly deal with finite fans in the categories **ARS** and **RS**. Our main result is Theorem 3.11 —the isomorphism theorem for finite ARS-fans— which proves that, in this case, the order of specialization alone determines the isomorphism type. The proof depends on the notion of a “standard generating system” which we introduce in 3.4. \square

3.1 Notation and Reminder (a) Notation 1.8 for finite (ARS- and RS-)fans is used systematically in this section, adapted in a self-explanatory way; e.g., for $1 \leq k \leq j \leq n = \ell(X_F)$, L_k (or $L_k(X_F)$, if necessary), will stand for the level L_{I_k} , S_j^k for the set $S_{I_j}^{I_k}$, etc.

(b) Recall that the AOSs have a combinatorial geometric (matroid) structure; it was introduced in [D1] and [D2] for spaces of orders of fields, and later generalized to abstract order spaces in [Li]. In general, ARSs do not possess such a structure. Thus, combinatorial geometric notions such as *dependent set*, *independent set*, *basis*, *closed set*, *closure*, *dimension*, etc., will always refer to the above-mentioned combinatorial geometric structure, and apply only to AOSs. For the definition and the mutual relationships, in the general context of matroid theory, of combinatorial notions such as those just mentioned, the reader is referred to [Wh].

Since the combinatorial geometric structure of any AOS is isomorphic to that of a set of vectors in a (possibly infinite-dimensional) vector space over the two-element field \mathbb{F}_2 with the structure induced by linear dependence (cf. [D1], Thm. 3.1, p. 618), the notions above coincide with the corresponding notions over vector spaces. For example, a subset $A \subseteq X$ of an AOS $(X, G, -1)$ (G a group of exponent 2) is dependent iff there are pairwise distinct elements $g, g_1, \dots, g_r \in A$ ($r \geq 2$), such that $g = g_1 \cdot \dots \cdot g_r$ (as characters of G). Since functions in X send -1 to -1 , this functional identity can only hold if r is odd. Likewise, A is closed iff the product of any odd number of members of A belongs to A . \square

Warning. In this section the words *closed set* and *closure* are used only in the combinatorial geometric sense just defined. \square

Lemma 3.2 Let (X, F) be an ARS-fan (not necessarily finite). Let $J \subseteq I$ be in $\text{Spec}(F)$, and let $A \subseteq L_J$, $B \subseteq L_I$, be sets such that:

- (i) The unique \rightsquigarrow -successor in L_I of each $g \in A$ belongs to B .
- (ii) Every $h \in B$ has a unique \rightsquigarrow -predecessor in A .

Then, A dependent $\Rightarrow B$ dependent.

Proof. By assumption there are pairwise distinct elements $g, g_1, \dots, g_r \in A$ such that $g = g_1 \cdot \dots \cdot g_r$; as observed above, r is odd ≥ 3 . Let h, h_1, \dots, h_r be the unique successors of g, g_1, \dots, g_r , resp., in B coming from (i); thus, $g \rightsquigarrow h$ and $g_i \rightsquigarrow h_i$, for $i = 1, \dots, r$. By 1.6(a) we have $g = g_1 \cdot \dots \cdot g_r \rightsquigarrow h_1 \cdot \dots \cdot h_r$. Since $h_1 \cdot \dots \cdot h_r \in L_I$ (r is odd) and g has a unique \rightsquigarrow -successor in L_I , we get $h = h_1 \cdot \dots \cdot h_r$.

By assumption (ii), every element in A is the *unique* predecessor of an element in B . Since $g_i \neq g_j$, we get $h_i \neq h_j$ for $1 \leq i \neq j \leq r$; likewise, $h \neq h_i$ for $i = 1, \dots, r$. This proves that h is the product of r *distinct* elements in B , and hence that B is dependent. \square

Proposition 3.3 (Choice of basis). Let (X, F) be a finite ARS-fan; let $1 \leq k < n = \ell(X)$. Let \mathcal{G} be an arbitrary AOS-subfan of $L_{k+1} = L_{k+1}(X)$. Let $\mathcal{F} = \{h \in L_k \mid \text{There is } g \in \mathcal{G} \text{ such that } g \rightsquigarrow h\}$ be the AOS-fan consisting of the depth- k successors of elements of \mathcal{G} (cf. 2.5). Assume:

(*) $\forall h, h' \in \mathcal{F}, \text{card}(\{g \in \mathcal{G} \mid g \rightsquigarrow h\}) = \text{card}(\{g \in \mathcal{G} \mid g \rightsquigarrow h'\})$.

Let $\mathcal{B} = \{h_1, \dots, h_r\}$ be a basis of \mathcal{F} (as an AOS), and let \mathcal{C} be a basis of the AOS-fan $P_{h_1} = \{g \in \mathcal{G} \mid g \rightsquigarrow h_1\}$ (see 2.11 (1)). For $i = 2, \dots, r$, let $g_i \in \mathcal{G}$ be such that $g_i \rightsquigarrow h_i$.

Then, $\mathcal{C} \cup \{g_2, \dots, g_r\}$ is a basis of \mathcal{G} .

Proof. If $r = 1$, then $\mathcal{F} = \mathcal{B} = \{h_1\}$, whence $\mathcal{G} = \{g \in \mathcal{G} \mid g \rightsquigarrow h_1\}$, and the result holds by the choice of \mathcal{C} . Henceforth we assume $r \geq 2$. We observe:

— $r = \text{card}(\mathcal{B}) = \dim(\mathcal{F})$. Since \mathcal{F} is an AOS-fan, $\text{card}(\mathcal{F}) = 2^{r-1}$.

— For every $h \in \mathcal{F}$, $A_h = \{g \in \mathcal{G} \mid g \rightsquigarrow h\}$ is a AOS-fan; this follows from the assumption that \mathcal{G} is an AOS-fan, since A_h is closed under the product of any three of its elements, cf. Lemma 1.6 (b).

— $A_h \cap A_{h'} = \emptyset$ for $h \neq h'$ in \mathcal{F} .

By assumption (*), $\text{card}(A_h) = \text{card}(A_{h'}) (= 2^{p-1}$, say), for $h, h' \in \mathcal{F}$. Since $\mathcal{G} = \bigcup_{h \in \mathcal{F}} A_h$, we get $\text{card}(\mathcal{G}) = \text{card}(\mathcal{F}) \cdot \text{card}(A_h)$ (any $h \in \mathcal{F}$), and then $\text{card}(\mathcal{G}) = 2^{r-1} \cdot 2^{p-1} = 2^{p+r-2}$; hence $\dim(\mathcal{G}) = p + r - 1$. Since $\text{card}(\mathcal{C} \cup \{g_2, \dots, g_r\}) = p + r - 1$, it suffices to prove:

(**) $\mathcal{C} \cup \{g_2, \dots, g_r\}$ is an independent set.

Proof of (**). Assume false.

Case 1. Some g_{i_0} , with $2 \leq i_0 \leq r$, is dependent on the rest, i.e., there are $\mathcal{C}' \subseteq \mathcal{C}$ and $J \subseteq \{2, \dots, r\} \setminus \{i_0\}$ so that $g_{i_0} = \prod_{c \in \mathcal{C}'} c \cdot \prod_{j \in J} g_j$, i.e.,

(+) $\prod_{c \in \mathcal{C}'} c = \prod_{j \in J \cup \{i_0\}} g_j$.

— If $\text{card}(\mathcal{C}')$ is odd, since A_{h_1} is an AOS-fan, and hence a closed set, the left-hand side of (+) is an element $g' \rightsquigarrow h_1$, and we have $g' \cdot \prod_{j \in J \cup \{i_0\}} g_j = 1$. Setting $A = \{g'\} \cup \{g_j \mid j \in J \cup \{i_0\}\} \subseteq L_{k+1}$ and $B = \{h_1\} \cup \{h_j \mid j \in J \cup \{i_0\}\} \subseteq L_k$, the assumptions of Lemma 3.2 are met. Since A is dependent, so is B , contradicting that $B \subseteq \mathcal{B}$ and \mathcal{B} is a basis of \mathcal{F} , whence an independent set.

— If $\mathcal{C}' = \emptyset$, the same argument works, yielding a contradiction.

— Assume $\text{card}(\mathcal{C}')$ even > 0 . Fix $c_0 \in \mathcal{C}'$. Then $\text{card}(\mathcal{C}' \setminus \{c_0\})$ is odd, and $g' = \prod_{c \in \mathcal{C}' \setminus \{c_0\}} c \in L_{k+1}$; also $g' \rightsquigarrow h_1$, and we have:

$$c_0 \cdot g' \cdot \prod_{j \in J \cup \{i_0\}} g_j = 1.$$

Pick any index $j_0 \in J \cup \{i_0\}$ (so, $j_0 \geq 2$). Since $c_0, g' \rightsquigarrow h_1$ and $g_{j_0} \rightsquigarrow h_{j_0}$, Lemma 1.6(b) yields $g'_{j_0} := c_0 g' g_{j_0} \rightsquigarrow h_{j_0}$, and $g'_{j_0} \cdot \prod_{j \in (J \cup \{i_0\}) \setminus \{j_0\}} g_j = 1$. Hence, $A = \{g'_{j_0}\} \cup \{g_j \mid j \in (J \cup \{i_0\}) \setminus \{j_0\}\}$ is a dependent subset of L_{k+1} . Setting $B = \{h_j \mid j \in J \cup \{i_0\}\}$ the assumptions of Lemma 3.2 are met, and hence B is also dependent, contradicting that $B \subseteq \mathcal{B}$.

Case 2. Some $c_0 \in \mathcal{C}$ is dependent on the rest.

Then, there are $\mathcal{C}' \subseteq \mathcal{C} \setminus \{c_0\}$ and $J \subseteq \{2, \dots, r\}$ so that

$$(++) \quad c_0 = \prod_{c \in \mathcal{C}'} c \cdot \prod_{j \in J} g_j.$$

Note that $J \neq \emptyset$ (otherwise \mathcal{C} would be dependent). Taking J minimal so that $(++)$ holds, and picking $j_0 \in J$, it follows that c_0 is not in the closure of $\mathcal{C}' \cup \{g_j \mid j \in J \setminus \{j_0\}\}$ (cf. Warning, end of 3.1 (b)). By the exchange property, g_{j_0} is in the closure of $\mathcal{C}' \cup \{c_0\} \cup \{g_j \mid j \in J \setminus \{j_0\}\}$, contrary to the result of Case 1. \square

3.4 Standard generating systems.

For any finite ARS-fan, (X, F) , we will construct, by induction on k , $1 \leq k \leq n = \ell(X)$, a class of bases \mathcal{B}_k of the AOS-fan $L_k(X)$. Each basis \mathcal{B}_k will be required to satisfy the additional condition:

(*) For $k \leq j \leq n$, $\mathcal{B}_k \cap S_j^k$ is a basis of the AOS-fan S_j^k .

This additional requirement guarantees that the inductive construction of the \mathcal{B}_k 's is not interrupted before the n -th (and last) step. The construction uses Proposition 3.3 and the results from §2 above. The set $\mathcal{B} = \bigcup_{k=1}^n \mathcal{B}_k$ is called a **standard generating system** for (X, F) .

Construction of standard generating systems.

Level 1. It suffices to observe that a basis \mathcal{B}_1 of L_1 satisfying condition (*) exists. Begin by choosing a basis $\mathcal{B}_1(n)$ of the AOS-fan $S_n^1 = C_n^1$ (cf. Corollary 2.5). S_n^1 is a closed subset (cf. Warning, end of 3.1 (b)) of the (AOS-)fan $S_{n-1}^1 = S_n^1 \cup C_{n-1}^1$; hence, $\mathcal{B}_1(n)$ is an independent subset of S_{n-1}^1 ; choose $\mathcal{B}_1(n-1)$ to be a basis of S_{n-1}^1 extending $\mathcal{B}_1(n)$.

In general, assume that, for $1 < j \leq n$ an increasing sequence $\mathcal{B}_1(n) \subseteq \dots \subseteq \mathcal{B}_1(j)$ of independent subsets of L_1 has been chosen so that $\mathcal{B}_1(\ell)$ is a basis of the AOS-fan S_ℓ^1 ($j \leq \ell \leq n$). As above, $\mathcal{B}_1(j)$ is an independent subset of the fan $S_{j-1}^1 = S_j^1 \cup C_{j-1}^1$. Let $\mathcal{B}_1(j-1)$ be a basis of S_{j-1}^1 extending $\mathcal{B}_1(j)$. Set $\mathcal{B}_1 = \mathcal{B}_1(1)$; by construction, $\mathcal{B}_1 \cap S_j^1 = \mathcal{B}_1(j)$ is a basis of S_j^1 .

Induction step. Given an integer k , $1 \leq k < n$, assume there exists a basis \mathcal{B}_k of L_k satisfying property (*); thus, for $k \leq j \leq n$, $\mathcal{B}_k(j) = \mathcal{B}_k \cap S_j^k$ is a basis of S_j^k . Further, since $S_n^k \subseteq \dots \subseteq S_k^k = L_k$, we have $\mathcal{B}_k(n) \subseteq \dots \subseteq \mathcal{B}_k(k) = \mathcal{B}_k$. Using Proposition 3.3 we define a subset \mathcal{B}_{k+1} of L_{k+1} as follows:

— Firstly, fix an element $h_0 \in \mathcal{B}_k(n)$ (this set is non-empty because $n = \ell(X)$). Pick a basis $\mathcal{B}_{k+1}(n, h_0)$ of the AOS-fan $\{g \in S_n^{k+1} \mid g \rightsquigarrow h_0\}$.

— Next, for each $h \in (\mathcal{B}_k \cap S_{k+1}^k) \setminus \{h_0\}$ there is a maximal index $j = j(h)$, $k+1 \leq j \leq n$, so that $h \in \mathcal{B}_k \cap S_j^k = \mathcal{B}_k(j)$; clearly, $h \notin S_{j+1}^k$, whence $h \in C_j^k = S_j^k \setminus S_{j+1}^k$ (if $j = n$, then $h \in S_n^k = C_n^k$). Since $j \geq k+1$, we have $\{g \in C_j^{k+1} \mid g \rightsquigarrow h\} \neq \emptyset$. Choose an element $g_h \in C_j^{k+1}$ so that $g_h \rightsquigarrow h$.

— Finally, set

$$[*] \quad \mathcal{B}_{k+1} = \mathcal{B}_{k+1}(n, h_0) \cup \{g_h \mid h \in (\mathcal{B}_k \cap S_{k+1}^k) \setminus \{h_0\}\}.$$

Claim. For $k+1 \leq p \leq n$, $\mathcal{B}_{k+1} \cap S_p^{k+1}$ is a basis of S_p^{k+1} .

Proof of Claim. We apply Proposition 3.3 with the following choice of parameters:

- $\mathcal{G} = S_p^{k+1}$ (whence $\mathcal{F} = S_p^k$, since $k+1 \leq p$);
- $\mathcal{B} = \mathcal{B}_k \cap S_p^k$ (a basis of \mathcal{F});
- $\mathcal{C} = \mathcal{B}_{k+1}(n, h_0)$ (a basis of $\{g \in S_n^{k+1} \mid g \rightsquigarrow h_0\}$).

Proposition 2.10 (a) shows that the cardinality assumption

$$\text{card}(\{g \in S_p^{k+1} \mid g \rightsquigarrow h\}) = \text{card}(\{g \in S_p^{k+1} \mid g \rightsquigarrow h'\}), \quad (h, h' \in S_j^k)$$

of 3.3 holds. We conclude that

$$\mathcal{D} := \mathcal{B}_{k+1}(n, h_0) \cup \{g_h \mid h \in (\mathcal{B}_k \cap S_p^k) \setminus \{h_0\}\}$$

is a basis of S_j^{k+1} . The Claim follows from:

$$(\dagger) \quad \mathcal{B}_{k+1} \cap S_p^{k+1} = \mathcal{D}.$$

Proof of (\dagger) . Since $\mathcal{B}_{k+1}(n, h_0) \subseteq \mathcal{D} \cap \mathcal{B}_{k+1}$ (see [*]), we need only prove:

$$(\subseteq) \quad \text{If } h \in (\mathcal{B}_k \cap S_{k+1}^k) \setminus \{h_0\} \text{ and } g_h \in S_p^{k+1}, \text{ then } h \in \mathcal{B}_k \cap S_p^k.$$

This clearly follows from $g_h \in S_p^{k+1}$, $g_h \rightsquigarrow h$ and $h \in S_{k+1}^k$.

(\supseteq) Since $k+1 \leq p$, we have $\mathcal{B}_k \cap S_p^k = \mathcal{B}_k(p) \subseteq \mathcal{B}_k(k+1) = \mathcal{B}_k \cap S_{k+1}^k$. On the other hand, if $h \in (\mathcal{B}_k \cap S_p^k) \setminus \{h_0\}$ and, as above, $j(h)$ denotes the largest index j so that $k+1 \leq j \leq n$ and $h \in \mathcal{B}_k(j)$, we have $p \leq j(h)$, whence $S_{j(h)}^{k+1} \subseteq S_p^{k+1}$. By choice, $g_h \in C_{j(h)}^{k+1}$; it follows that $g_h \in S_p^{k+1}$, as required. \square

Remarks 3.5 (a) In general, there are many different standard generating systems for a finite ARS-fan (X, F) . The construction in 3.4 allows for several choices of the bases $\mathcal{B}_1(j)$ ($1 \leq j \leq n$) and, at each successive step, k , for many choices of elements $h_0 \in \mathcal{B}_k(n)$, of bases $\mathcal{B}_{k+1}(n, h_0)$, and of elements $g_h \in C_n^{k+1}$ under each $h \in (\mathcal{B}_k \cap S_{k+1}^k) \setminus \{h_0\}$. In spite of this lack of uniqueness, we shall prove below that any standard generating system determines the isomorphism type of a finite ARS-fan.

(b) Some of the sets $C_j^k = S_j^k \setminus S_{j+1}^k$ may be empty. However, if $C_j^k \neq \emptyset$, then, necessarily, $\mathcal{B}_k \cap C_j^k \neq \emptyset$. Indeed, if $j = n$, then $C_n^k \neq \emptyset$ (as $n = \ell(X)$) and $C_n^k = S_n^k$ is an AOS-fan; since $\mathcal{B}_k \cap C_n^k$ is a basis of C_n^k , it must contain at least one element. If $j < n$, since S_{j+1}^k is a fan, it is a closed set; as it is disjoint from C_j^k , then no element of C_j^k is dependent on S_{j+1}^k . Hence, any basis of $S_j^k = S_{j+1}^k \cup C_j^k$ must contain an element of C_j^k . \square

Any standard generating system for a finite ARS-fan has the following property:

Corollary 3.6 *Let \mathcal{B} be a standard generating system for a finite ARS-fan (X, F) ; let $n = \ell(X)$, and $1 \leq k \leq m \leq n$. Then, for every $g \in \mathcal{B}_m = \mathcal{B} \cap L_m(X)$, the unique depth- k successor of g in X belongs to \mathcal{B} (hence to $\mathcal{B}_k = \mathcal{B} \cap L_k(X)$).*

Proof. By the construction in 3.4 this holds for $m = k+1$. In fact, let $g \in \mathcal{B}_{k+1}$, $h' \in L_k$ and $g \rightsquigarrow h'$. By the definition of \mathcal{B}_{k+1} and uniqueness of the successor of g in L_k , if $g \in \mathcal{B}_{k+1}(n, h_0)$, then $h' = h_0 \in \mathcal{B}_k(n) \subseteq \mathcal{B}_k$; if $g = g_h$, with $h \in (\mathcal{B}_k \cap S_{k+1}^k) \setminus \{h_0\}$, we get $h' = h \in \mathcal{B}_k$. Then iterate. \square

For the proof of the Isomorphism Theorem 3.11 below we shall need the characterizations of ARS-morphisms between fans proved in 3.9 and 3.10 below, which, in turn, follow from the Small Representation Theorem 3.8.

Definition 3.7 Let G, H be RSs, let X_G, X_H be their character spaces, and let $Z \subseteq X_G$. A map $F : Z \rightarrow X_H$ **preserves 3-products** (in Z) iff for all $h_1, h_2, h_3 \in Z$,

$$h_1 h_2 h_3 \in Z \Rightarrow F(h_1 h_2 h_3) = F(h_1) F(h_2) F(h_3). \quad \square$$

Proposition 3.8 (Small representation theorem). *Let G be a RS. The following conditions are equivalent for a map $f : X_G \rightarrow \mathbf{3}$:*

(1) a) f is continuous in the constructible topology of X_G .

b) f preserves 3-products in X_G .

(2) f is represented by an element of G : there is $a \in G$ so that $f = \hat{a}$.

[$\hat{a} : X_G \rightarrow \mathbf{3}$ denotes “evaluation at a ”: for $h \in X_G$, $\hat{a}(h) = h(a)$.]

Proof. (2) \Rightarrow (1) is clear since the evaluation maps have properties (1.a) and (1.b).

(1) \Rightarrow (2). We use the representation theorem [M], Cor. 8.3.6, p. 162. It suffices to check that the assumptions of this theorem as well as one of the equivalent conditions in its conclusion hold under our hypotheses in (1). In our notation, the conditions to be checked are: for $x, y \in X_G$,

(†) $f(x) = 0$ and $Z(x) \subseteq Z(y)$ implies $f(y) = 0$.

(††) $f(x) \neq 0$ and $x^{-1}[0, 1] \supseteq y^{-1}[0, 1]$ implies $f(x) = f(y)$.

(†††) For any saturated prime ideal I of G , either

(i) $f[\{u \in X_G \mid Z(u) = I\}] = 0$, or

(ii) $\prod_{i=1}^4 f(x_i) = 1$ for any 4-element AOS-fan $\{x_1, \dots, x_4\}$ in $\{u \in X_G \mid Z(u) = I\}$.

— Condition (†) follows at once from Lemma 0.2 (2) (as $Z(x) \subseteq Z(y) \Rightarrow y = yx^2$).

— Condition (††) follows from Lemma 0.1 (3), (5): $x^{-1}[0, 1] \supseteq y^{-1}[0, 1] \Rightarrow x = x^2y$. Since $f(x) \neq 0 \Rightarrow f(x^2) = 1$, assumption (1.b) implies $f(x) = f(x^2)f(y) = f(y)$.

— As for (†††), if (i) does not hold, (†) implies $f(u) \neq 0$ for all $u \in X_G$ such that $Z(u) = I$. Let $\{x_1, \dots, x_4\}$ be an AOS-fan in $\{u \in X_G \mid Z(u) = I\}$. Thus, $x_4 = x_1 x_2 x_3$ and $f(x_i) \neq 0$ for $i = 1, \dots, 4$. Assumption (1.b) gives $f(x_4) = f(x_1)f(x_2)f(x_3) \neq 0$, i.e., $\prod_{i=1}^4 f(x_i) = 1$. \square

Corollary 3.9 *A map $F : (X_1, F_1) \rightarrow (X_2, F_2)$ between ARS-fans is an ARS-morphism iff F is continuous for the constructible topology (of both source and target) and preserves 3-products in X_1 (cf. 3.7).*

Proof. (\Leftarrow) If F has the stated properties and $a \in F_2$, then $\hat{a} \circ F : X_1 \rightarrow \mathbf{3}$ also has those properties, and, by Proposition 3.8, is represented by an element of F_1 ; hence, F is an ARS-morphism (cf. 1.1 (c.i)).

(\Rightarrow) Assume F is an ARS-morphism. For continuity it suffices to show that $F^{-1}[V]$ is open constructible in X_1 whenever V is basic open for the constructible topology of X_2 , i.e., of the form $V = U(a_1, \dots, a_n) \cap Z(a)$ with $a, a_1, \dots, a_n \in F_2$ (see [M], p. 111). By the assumption on F , there are $b, b_1, \dots, b_n \in F_1$ such that $\hat{a} \circ F = \hat{b}$ and $\hat{a}_i \circ F = \hat{b}_i$ for $i = 1, \dots, n$. These functional identities imply $F^{-1}[V] = U(b_1, \dots, b_n) \cap Z(b)$, as required.

Preservation of 3-products by F follows easily from the same property for \hat{a} and \hat{b} using the functional identity $\hat{a} \circ F = \hat{b}$. \square

Lemma 3.10 *Let $(X_1, F_1), (X_2, F_2)$ be ARS-fans.*

- (1) For a map $F : X_1 \longrightarrow X_2$ the following are equivalent:
- (i) F preserves 3-products in X_1 .
 - (ii) a) F preserves 3-products of elements of the same level: for all $I \in \text{Spec}(F_1)$ and all $h_1, h_2, h_3 \in L_I(X_1)$, $F(h_1 h_2 h_3) = F(h_1)F(h_2)F(h_3)$.
b) F is monotone for the specialization order: for $g, h \in X_1$, $g \rightsquigarrow_1 h \Rightarrow F(g) \rightsquigarrow_2 F(h)$.
(\rightsquigarrow_i denotes specialization in X_i).
- (2) If (X_1, F_1) is finite, any map verifying one of the equivalent conditions (i) or (ii) in (1) is a morphism of ARSs.
- (3) If both (X_1, F_1) , (X_2, F_2) are finite, any bijection $F : X_1 \longrightarrow X_2$ verifying one of the equivalent conditions in (1) is an isomorphism of ARSs.

Proof. (1). (i) \Rightarrow (ii). (ii.a) is a special case of (i).

(ii.b) $g \rightsquigarrow_1 h \Leftrightarrow h = h^2 g$ (Lemma 0.1). By (i), $F(h) = F(h)^2 F(g)$, and this equality (in X_2) gives $F(g) \rightsquigarrow_2 F(h)$.

(ii) \Rightarrow (i). Let h_1, h_2, h_3 be any three elements in X_1 ; say $Z(h_3) \subseteq Z(h_2) \subseteq Z(h_1)$. Let $I = Z(h_1)$ and for $i = 2, 3$ let h'_i be the unique successor of h_i in $L_I(X_1)$; Lemma 1.5 shows that $h_1 h_2 h_3 = h_1 h'_2 h'_3$; then, assumption (ii.a) gives

$$F(h_1 h_2 h_3) = F(h_1)F(h'_2)F(h'_3).$$

By (ii.b) we have $F(h_i) \rightsquigarrow_2 F(h'_i)$, ($i = 2, 3$). Next, note that $Z(F(h'_i)) \subseteq Z(F(h_1))$ for $i = 2, 3$. In fact, since h_1, h'_2 belong to the same level L_I , $Z(h_1) = Z(h'_2)$, and Lemma 0.2 (2) yields $h_1^2 = h'^2_2$; scaling by h_1 gives $h_1 = h_1 h'^2_2$. Since F preserves 3-products of the same level, $F(h_1) = F(h_1)F(h'_2)^2$ which, by 0.2 (1), gives $Z(F(h'_2)) \subseteq Z(F(h_1))$. Same argument for $i = 3$. Using 1.5 again, these inclusions and $F(h_i) \rightsquigarrow_2 F(h'_i)$, ($i = 2, 3$) prove:

$$F(h_1)F(h'_2)F(h'_3) = F(h_1)F(h_2)F(h_3),$$

as required.

(2) follows at once from Corollary 3.9, since the continuity requirement is automatically fulfilled in this case: the constructible topology in X_1 is discrete.

(3) By (2) it only remains to prove that $F^{-1} : X_2 \longrightarrow X_1$ preserves 3-products in X_2 . Let $g_1, g_2, g_3 \in X_2$ and let $h_i = F^{-1}(g_i)$, $i = 1, 2, 3$. From (1.i) we have $F(h_1 h_2 h_3) = g_1 g_2 g_3$. Composing both sides of this equality with F^{-1} gives the desired conclusion:

$$F^{-1}(g_1 g_2 g_3) = F^{-1}(F(h_1 h_2 h_3)) = h_1 h_2 h_3 = F^{-1}(g_1)F^{-1}(g_2)F^{-1}(g_3). \quad \square$$

Remark. Note that any isomorphism of ARS-fans preserves depth.

Theorem 3.11 (The isomorphism theorem for finite ARS-fans.) *Let (X_1, F_1) , (X_2, F_2) be finite ARS-fans and let $\rightsquigarrow_1, \rightsquigarrow_2$ denote their respective specialization orders. If $(X_1, \rightsquigarrow_1)$ and $(X_2, \rightsquigarrow_2)$ are order-isomorphic, then X_1 and X_2 are isomorphic ARSs.*

Proof. The order-isomorphism assumption implies:

(1) $\ell(X_1) = \ell(X_2)$ ($= n$, say, fixed throughout the proof).

(2) For $1 \leq k \leq j \leq n$, $\text{card}(C^k_j(X_1)) = \text{card}(C^k_j(X_2))$.

The proof of (2) is an easy exercise. Since $C^k_\ell \cap C^k_{\ell'} = \emptyset$ for $k \leq \ell \neq \ell' \leq n$ and $S^k_j = \bigcup_{\ell=j}^n C^k_\ell$, we get:

(3) For $1 \leq k \leq j \leq n$, $\text{card}(S^k_j(X_1)) = \text{card}(S^k_j(X_2))$.

(4) For $1 \leq k < n$ and all $h \in S^k_n(X_1)$, $h' \in S^k_n(X_2)$, we have:

$$\text{card}(\{g \in S_n^{k+1}(X_1) \mid g \rightsquigarrow_1 h\}) = \text{card}(\{g' \in S_n^{k+1}(X_2) \mid g' \rightsquigarrow_2 h'\}).$$

Proof of (4). Consider the two-variable formula in the language $\{\leq\}$ of order:

$$\varphi(x, y) := x \in S_n^{k+1} \wedge x \leq y.$$

(It is left as an exercise for the reader to write a first-order formula in $\{\leq\}$ expressing the notion $x \in S_n^{k+1}$; cf. 2.1.)

If σ denotes the order isomorphism between $(X_1, \rightsquigarrow_1)$ and $(X_2, \rightsquigarrow_2)$, for $g, h \in X_1$ we have:

$$(X_1, \rightsquigarrow_1) \models \varphi[g, h] \Leftrightarrow (X_2, \rightsquigarrow_2) \models \varphi[\sigma(g), \sigma(h)].$$

It follows that σ maps $\{g \in S_n^{k+1}(X_1) \mid g \rightsquigarrow_1 h\}$ bijectively onto $\{g' \in S_n^{k+1}(X_2) \mid g' \rightsquigarrow_2 \sigma(h)\}$.

Now, if $h \in S_n^k(X_1)$, then $\sigma(h) \in S_n^k(X_2)$. If $h' \in S_n^k(X_2)$, apply Proposition 2.10 with $h_1 = h'$ and $h_2 = \sigma(h)$ to conclude. \square

Since the sets in item (4) are AOS-fans (Corollary 2.5), they have the same dimension, i.e., any bases of each of them have the same cardinality. If $\mathcal{B}^1, \mathcal{B}^2$ are standard generating systems for X_1, X_2 , respectively, then $\mathcal{B}^i \cap S_j^k(X_i)$ is a basis of the fan $S_j^k(X_i)$, for $1 \leq k \leq j \leq n$ and $i = 1, 2$; from (3) we get:

$$(5) \text{ For } 1 \leq k \leq j \leq n, \text{ card}(\mathcal{B}^1 \cap S_j^k(X_1)) = \text{card}(\mathcal{B}^2 \cap S_j^k(X_2)).$$

In particular, for $S_k^k = L_k$ we obtain:

$$(6) \text{ If } 1 \leq k \leq n, \text{ then } \text{card}(\mathcal{B}_k^1) = \text{card}(\mathcal{B}_k^2) \text{ (where } \mathcal{B}_k^i = \mathcal{B}^i \cap L_k(X_i)).$$

Next, we choose an arbitrary standard generating system \mathcal{B}^1 for X_1 . By induction on k , $1 \leq k \leq n$, we construct a standard generating system \mathcal{B}^2 of X_2 ($\mathcal{B}^2 = \bigcup_{k=1}^n \mathcal{B}_k^2$) and a map $f_k : \mathcal{B}_k^1 \rightarrow \mathcal{B}_k^2$ so that:

$$(7) \text{ i) For } k \leq j \leq n, f_k[\mathcal{B}^1 \cap S_j^k(X_1)] = \mathcal{B}^2 \cap S_j^k(X_2).$$

$$\text{ii) If } k < n, g \in \mathcal{B}_{k+1}^1, h \in \mathcal{B}_k^1 \text{ and } g \rightsquigarrow_1 h, \text{ then } f_{k+1}(g) \rightsquigarrow_2 f_k(h).$$

Construction of \mathcal{B}^2 and the maps f_k .

Level 1. \mathcal{B}_1^2 is built as in the level 1 step in 3.4; with notation therein, $f_1 : \mathcal{B}_1^1 \rightarrow \mathcal{B}_1^2$ is any bijection mapping $\mathcal{B}_1^1(j)$ onto $\mathcal{B}_2^2(j)$, for $1 \leq j \leq n$. Such a bijection exists by (5) above ($k = 1$).

Induction step. Assume $\mathcal{B}_1^2, \dots, \mathcal{B}_k^2$ and f_1, \dots, f_k already constructed, so that:

— For $1 \leq j \leq k$ and $j \leq \ell \leq n$, $\mathcal{B}_j^2 \cap S_\ell^j(X_2)$ is a basis of the AOS-fan $S_\ell^j(X_2)$ and $f_j[\mathcal{B}_j^1 \cap S_\ell^j(X_1)] = \mathcal{B}_j^2 \cap S_\ell^j(X_2)$.

— Condition (7.ii) holds for all j such that $1 \leq j < k$.

The basis \mathcal{B}_{k+1}^2 , and along with it the map f_{k+1} , are defined by performing the construction of the inductive step in 3.4, with the following choice of parameters:

— If $h_0 \in \mathcal{B}_k^1 \cap S_n^k(X_1)$, and $\mathcal{B}_{k+1}^1(n, h_0)$ is a basis of the (AOS-)fan $\{g \in S_n^{k+1}(X_1) \mid g \rightsquigarrow_1 h_0\}$, then take $\mathcal{B}_{k+1}^2(n, f_k(h_0))$ to be a basis of the fan $\{g' \in S_n^{k+1}(X_2) \mid g' \rightsquigarrow_2 f_k(h_0)\}$. This is possible since $f_k(h_0) \in \mathcal{B}_k^2 \cap S_n^k(X_2)$, by (7.i). Using item (4), we let $f_{k+1} \upharpoonright \mathcal{B}_{k+1}^1(n, h_0)$ be a bijection between $\mathcal{B}_{k+1}^1(n, h_0)$ and $\mathcal{B}_{k+1}^2(n, f_k(h_0))$.

— If $g \in \mathcal{B}_{k+1}^1 \cap C_j^{k+1}(X_1)$ with $k+1 \leq j \leq n$, but $g \notin \mathcal{B}_{k+1}^1(n, h_0)$, then, by the construction performed in the inductive step of 3.4, if h is the unique depth- k successor of g , we have $h \in \mathcal{B}_k^1 \cap C_j^k(X_1)$, $h \neq h_0$ and $g = g_h$. In this case choose any element $g' \rightsquigarrow_2 f_k(h)$ such that $g' \in C_j^{k+1}(X_2)$, and set $f_{k+1}(g) = g'$. This is possible since $f_k(h) \in \mathcal{B}_k^2 \cap C_j^k(X_2)$ (which follows

easily from (7.i)). Clearly, this construction guarantees that (7.i) and (7.ii) hold for $k + 1$.

Note that (7.ii) implies, by iteration, its own generalization:

(7) iii) If $1 \leq k < m \leq n$, $g \in \mathcal{B}_m^1$, $h \in \mathcal{B}_k^1$ and $g \rightsquigarrow_1 h$, then $f_m(g) \rightsquigarrow_2 f_k(h)$.

Since $\mathcal{B}_k^i = \mathcal{B}^i \cap L_k(X_i)$ is a basis of the AOS-fan $L_k(X_i)$, $i = 1, 2$, we get:

(8) The bijection f_k extends (uniquely) to an AOS-isomorphism $\tilde{f}_k : L_k(X_1) \longrightarrow L_k(X_2)$ mapping $S_j^k(X_1)$ onto $S_j^k(X_2)$, for all j such that $k \leq j \leq n$.

Now set $F : X_1 \longrightarrow X_2$ to be $F = \bigcup_{k=1}^n \tilde{f}_k$. We prove:

Claim. F is an isomorphism of ARSs.

Proof of Claim. Since $X_i = \bigcup_{k=1}^n L_k(X_i)$ (disjoint union) for $i = 1, 2$, and \tilde{f}_k maps $L_k(X_1)$ bijectively onto $L_k(X_2)$, we have:

(a) F is well-defined and bijective.

(b) For all k , $1 \leq k \leq n$, F preserves 3-products in L_k .

This is clear: by (8) $F|_{L_k(X_1)} = \tilde{f}_k : L_k(X_1) \longrightarrow L_k(X_2)$ is an isomorphism of AOS-fans.

(c) F is monotone for the specialization order.

Let $g, h \in X_1$ be such that $g \rightsquigarrow_1 h$; say $d(g) = m \geq d(h) = k$. We must prove $F(g) \rightsquigarrow_2 F(h)$. Since \mathcal{B}_m^1 generates $L_m(X_1)$, then $g = g_1 \cdot \dots \cdot g_r$ with $g_1, \dots, g_r \in \mathcal{B}_m^1$ and r necessarily odd (possibly = 1). By Corollary 3.6, if h_i is the unique depth- k successor of g_i , then $h_i \in \mathcal{B}_k^1$. Also, $g_i \rightsquigarrow_1 h_i$ ($i = 1, \dots, r$) implies $g = g_1 \cdot \dots \cdot g_r \rightsquigarrow_1 h_1 \cdot \dots \cdot h_r$ (1.6 (a)). Since both h and $h_1 \cdot \dots \cdot h_r$ are successors of g of the same level k , we get $h = h_1 \cdot \dots \cdot h_r$. As F preserves products of any odd number of elements of the same level, we have:

$$F(g) = F(g_1) \cdot \dots \cdot F(g_r) \quad \text{and} \quad F(h) = F(h_1) \cdot \dots \cdot F(h_r).$$

Since $g_i \rightsquigarrow_1 h_i$, $g_i \in \mathcal{B}_m^1$ and $h_i \in \mathcal{B}_k^1$, item (7.iii) yields $F(g_i) = f_m(g_i) \rightsquigarrow_2 f_k(h_i) = F(h_i)$ ($i = 1, \dots, r$). Then, by 1.6 (a) again,

$$F(g) = F(g_1) \cdot \dots \cdot F(g_r) \rightsquigarrow_2 F(h_1) \cdot \dots \cdot F(h_r) = F(h),$$

which proves (c). The Claim follows from (a)–(c) using Lemma 3.10 (3). This completes the proof of Theorem 3.11. \square

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